# Maximization Assignment Problem :-

The problem may be to assign persons to the jobs in such a way that the expected profit is maximized. Such maximization problem may be solved by converting it to minimization problem, This is done by converting the profit matrix to the cost (loss) matrix in either of the following two ways.

i) Subtract each element of the given matrix (Profit Matrix) from the greatest element of the matrix to get the equivalent cost (i.e loss) matrix.

ii) Place minus sign before each element of the profit matrix to get the equivalent cost matrix.

**eg:-** From the following data, obtain which product each plant should produce to maximize profit ?

Sales (revenue) Rs (1000)

| Plant↓ | | | | |
|---|---|---|---|---|
| A | 50 | 68 | 49 | 62 |
| B | 60 | 70 | 51 | 74 |
| C | 55 | 67 | 53 | 70 |
| D | 58 | 65 | 54 | 69 |

Production Cost Rs (1000)

| Plant↓ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 49 | 60 | 45 | 61 |
| B | 55 | 63 | 45 | 69 |
| C | 52 | 62 | 49 | 68 |
| D | 55 | 64 | 48 | 66. |

Since Profit = Sales revenue − Production Cost, so the profit matrix is

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 1 | 8 | 4 | 1 |
| B | 5 | 7 | 6 | 5 |
| C | 3 | 5 | 4 | 2 |
| D | 3 | 1 | 6 | 3 |

→ Maximization Problem,

**Method:1** Subtract each element of the above Matrix from the greatest element 8 we get.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 7 | 0 | 4 | 7 |
| B | 3 | 1 | 2 | 3 |
| C | 5 | 3 | 4 | 6 |
| D | 5 | 7 | 2 | 5 |

Now subtract the minimum value of corresponding row, we get.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 7 | 0 | 4 | 7 |
| B | 2 | 0 | 1 | 2 |
| C | 2 | 0 | 1 | 3 |
| D | 3 | 5 | 0 | 3. |

Now subtract the minimum value to the corresponding column.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 5 | $\boxed{0}$ | 3 | 5 |
| B | 0 | ~~0~~ | ~~0~~ | $\boxed{0}$ |
| C | $\boxed{0}$ | ~~0~~ | ~~0~~ | 1 |
| D | 1 | 5 | $\boxed{0}$ | 1 |

A → 2, B → 4, C → 1, D → 3.

**Method: 2.**

Place negative sign before each element we get

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | -1 | -8 | -4 | -1 |
| B | -5 | -7 | -6 | -5 |
| C | -3 | -5 | -4 | -2 |
| D | -3 | -1 | -6 | -3 |

Now subtract the minimum value from each row, we get

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 7 | 0 | 4 | 7 |
| B | 2 | 0 | 1 | 2 |
| C | 2 | 0 | 1 | 3 |
| D | 3 | 5 | 0 | 3 |

Now subtract the minimum value from each colum, we get

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 5 | $\boxed{0}$ | 4 | 5 |
| B | 0 | ~~0~~ | 1 | $\boxed{0}$ |
| C | $\boxed{0}$ | ~~0~~ | 1 | 1 |
| D | 1 | 5 | $\boxed{0}$ | 1 |

A → 2
B → 4
C → 1
D → 3.

Ques A Company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows (hundreds of rupees)

Job

| Machine | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 2.5 | 5 | 1 | 6 | 1 |
| 2 | 2 | 5 | 1.5 | 7 | 3 |
| 3 | 3 | 6.5 | 2 | 8 | 3 |
| 4 | 3.5 | 7 | 2 | 9 | 4.5 |
| 5 | 4 | 7 | 3 | 9 | 6 |
| 6 | 6 | 9 | 5 | 10 | 6 |

Solⁿ The given matrix is not a square matrix, we add one fictitious job 6 (Column) to make it a square matrix

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2.5 | 5 | 1 | 6 | 1 | 0 |
| 2 | 2 | 5 | 1.5 | 7 | 3 | 0 |
| 3 | 3 | 6.5 | 2 | 8 | 3 | 0 |
| 4 | 3.5 | 7 | 2 | 9 | 4.5 | 0 |
| 5 | 4 | 7 | 3 | 9 | 6 | 0 |
| 6 | 6 | 9 | 5 | 10 | 6 | 0 |

Now subtract the smallest element of each row from every element of the corresponding row, and then subtracting smallest element of each column from every element of the corresponding column, we get

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | .5 | $\boxed{0}$ | 0 | 0 | 0 | 0 |
| 2 | $\boxed{0}$ | 0 | .5 | 1 | 2 | 0 |
| 3 | 1 | 1.5 | 1 | 2 | 2 | $\boxed{0}$ |
| 4 | 1.5 | 2 | 1 | 3 | 3.5 | 0 |
| 5 | 2 | 2 | 2 | 3 | 5 | 0 |
| 6 | 4 | 4 | 4 | 4 | 5 | 0 |

The smallest element among the uncovered elements is 1. So subtracting this element 1 from all the uncovered elements and adding 1 to 6 column

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | .5 | $\cancel{0}$ | $\cancel{0}$ | $\boxed{\cancel{0}}$ | 0 | 0 |
| 2 | $\cancel{0}$ | $\boxed{0}$ | .5 | 1 | 2 | 0 |
| 3 | $\boxed{0}$ | .5 | $\boxed{\cancel{0}}$ | 1 | 1 | $\cancel{0}$ |
| 4 | .5 | 1 | $\boxed{0}$ | 2 | 2.5 | $\cancel{0}$ |
| 5 | 1 | 1 | 1 | 2 | 4 | $\boxed{0}$ |
| 6 | 3 | 3 | 3 | 3 | 4 | 0 |

Again subtracting the smallest element 1 from the uncovered elements and add to the rem. elements of 6 column.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | .5 | $\cancel{0}$ | $\cancel{0}$ | $\boxed{0}$ | $\cancel{0}$ | 2 |
| 2 | $\cancel{0}$ | $\cancel{0}$ | .5 | 1 | 2 | 2 |
| 3 | $\boxed{0}$ | .5 | $\cancel{0}$ | 1 | 1 | 1 |
| 4 | .5 | 1 | $\boxed{0}$ | 2 | 2.5 | 1 |
| 5 | $\cancel{0}$ | $\boxed{0}$ | $\cancel{0}$ | 1 | 2 | $\cancel{0}$ |
| 6 | 2 | 2 | 2 | 2 | 3 | $\boxed{0}$ |

Now again subtracting the smallest element from the uncovered elements and adding that lies the intersection of two lines and leaving remaining

Case II

~~1 → 5~~

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1.5 | 1 | 1 | $\boxed{0}$ | $\cancel{0}$ | 3 |
| 2 | $\boxed{0}$ | $\cancel{0}$ | .5 | $\cancel{0}$ | 1 | 2 |
| 3 | $\cancel{0}$ | .5 | $\boxed{\cancel{0}}$ | $\cancel{0}$ | $\boxed{0}$ | 1 |
| 4 | .5 | 1 | $\boxed{\cancel{0}}$ | 1 | 1.5 | 1 |
| 5 | $\cancel{0}$ | 0 | $\cancel{0}$ | $\cancel{0}$ | 1 | $\boxed{0}$ |
| 6 | 2 | 2 | 1 | 1 | 2 | $\boxed{0}$ |

Hence the optimal solution are,

Case I if 1 → 4

2 → 1 or 2 → 2

3 → 1 or 3 → 2 or ~~3 → 5~~ 3 → 5

4 → 3,

5 → 1 or 5 → 2 or 5 → 3

or ~~5 → 4~~ 5 → 5

6 → 6

## find an Initial feasible solution :-

### Method :1 North-West Corner Method :-

**Ques**.

| | $W_1$ | $W_2$ | $W_3$ | Supply |
|---|---|---|---|---|
| $f_1$ | 2 | 7 | 4 | 5 |
| $f_2$ | 3 | 3 | 1 | 8 |
| $f_3$ | 5 | 4 | 7 | 7 |
| $f_4$ | 1 | 6 | 2 | 14 |
| Demand | 7 | 9 | 18 | 34 |

→ Balanced.

**Sol^n Step:1** Start with the cell (1,1) at the north-west corner ie the top most left corner and allocate it maximum possible amount.

**Step:2** Then move to the right hand cell (1,2) if there is still any available quantity left, otherwise move to down cell (2,1) and allocate it maximum possible amount, and repeat this process until all the available quantity is exhausted.

| | $W_1$ | $W_2$ | $W_3$ | |
|---|---|---|---|---|
| $f_1$ | (5) 2 | 7 | 4 | 5/0 |
| $f_2$ | (2) 3 | (6) 3 | 1 | 8/6/0 |
| $f_3$ | 5 | (3) 4 | (4) 7 | 7/4/0 |
| $f_4$ | 1 | 6 | (14) 2 | 14/0 |
| | 7/2/0 | 9/3/0 | 18/14/0 | 34 |

$m+n-1$

$4+3-1 = 6$

No of allocation $= 6$,

$$\text{Total transportation Cost} = 5\times2 + 2\times3 + 6\times3 + 3\times4 + 4\times7 + 14\times2$$
$$= 10+6+18+12+28+28$$
$$= 16+30+56$$
$$= 16+86 = 102$$

### Method :2

### lowest Cost Entry Method :-

**Ques**.

| | $W_1$ | $W_2$ | $W_3$ | |
|---|---|---|---|---|
| $f_1$ | 2 | 7 | 4 | 5 |
| $f_2$ | 3 | 3 | 1 | 8 |
| $f_3$ | 5 | 4 | 7 | 7 |
| $f_4$ | 1 | 6 | 2 | 14 |
| | 7 | 9 | 18 | |

**Solution**:

**Step:1** Examine the cost matrix we find that there is lowest cost 1 in cell (2,3) and in (4,1) and allocate the maximum amount 8 to this cell which is more than the maximum amount 7 that can be allocated cell (4,1).

Continuing in this way we get the required feasible solution.

# Assignment Problem...

The assignment problem can be stated in the form of $n \times n$ matrix $[c_{ij}]$ called the cost or effectiveness matrix where $c_{ij}$ is the cost of assigning $i$-th facility (person) to the $j$-th job.

Effectiveness Matrix

Jobs.

$$
\text{Persons}\quad
\begin{array}{c|cccccc|}
 & 1 & 2 & 3 \cdots & j \cdots & & n \\
\hline
1 & c_{11} & c_{12} & c_{13} \cdots & c_{1j} \cdots & & c_{1n} \\
2 & c_{21} & c_{22} & c_{23} \cdots & c_{2j} \cdots & & c_{2n} \\
3 & \cdots & \cdots & \cdots & \cdots & & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\
i & c_{i1} & c_{i2} & c_{i3} & \cdots c_{ij} \cdots & & c_{in} \\
\vdots & & & & & & \\
n & c_{n1} & c_{n2} & c_{n3} & c_{nj} \cdots & & c_{nn} \\
\hline
\end{array}
$$

## Mathematically formulation of Assignment Problem

Mathematically an assignment problem can be stated as

Minimize the total cost

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

where $x_{ij} = \begin{cases} 1 & \text{if } i\text{-th person is assigned to } j^{th} \text{ job} \\ 0 & \text{if } i\text{-th person is not assigned to } j^{th} \text{ job.} \end{cases}$

Subject to the Conditions

i) $\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, 3 \ldots n$

Which means that only one job is done by the $i$-th person $i = 1, 2 \ldots n$.

ii) $\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2 \ldots n$

Which means that only one person is assigned to the $j$th job, $j = 1, 2 \ldots n$.

### Theorem :1 (Reduction Theorem)

Statement :- If in an assignment problem, a constant is added or subtracted to every element of a row (or column) of the cost matrix $[c_{ij}]$ then an assignment which minimize the total cost for one matrix, also minimizes the total cost for the other matrix,

If $x_{ij} = X_{ij}$ minimizes $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$ over all $x_{ij}$. st.

$\sum_{i=1}^{n} x_{ij} = 1 = \sum_{j=1}^{n} x_{ij}$ and $x_{ij} \geq 0$ then $x_{ij} = X_{ij}$ also minimizes $Z' = \sum_{i=1}^{n} \sum_{j=1}^{n} c'_{ij} \cdot x_{ij}$

where $c'_{ij} = c_{ij} \pm a_i \pm b_j$ ; $a_i, b_j$ are constant.

# Transportation Problem...

**Difference b/w a Transportation Problem or Assignment Problem :-**
An assignment Problem is a special case of the transportation problem in which $m=n$. and all the $a_i$'s and $b_j$'s ($i=1,2,\dots m$, $j=1,2\dots n$) are unity and each $x_{ij}$ is limited to one of the two values 0 and 1.
In these circumstances, exactly $n$ of $x_{ij}$ Can be non zero, one in each row of the array and one in each Column showing that only one source (person) can be assigned to each destination (job)

**A feasible Solution :-** A feasible solution to a transportation problem is a set of non negative individual allocation ($x_{ij} \geq 0$) which satisfies the row and column sum restrictions

**Basic feasible Solution :-** A feasible solution of m by n transportation problem is said to be a basic feasible solution if the total number of positive allocations $x_{ij}$ is exactly equal to $m+n-1$ ie one less than the sum of the numbers of rows and column.

**Optimal Solution :-** A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation Cost.

**Non-Degenerate basic feasible solution :-** If a f.s involves exactly $(m+n-1)$ independent individual positive allocation, then it is said to non degenerate B.F.S otherwise it is said to be degenerate B.F.S.

**Mathematical formulation of Transportation**
The transportation problem can be stated as follows,
find $x_{ij}$ ($i=1,2\dots m$, $j=1,2,\dots n$) for which the total transportation Cost

$$Z=\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \quad \text{--- (1)}$$

minimized subject to the restrictions

$$\left.\begin{aligned} \sum_{j=1}^{n} x_{ij} &= a_i \qquad i=1,2,3\dots m \\ \sum_{i=1}^{m} x_{ij} &= b_j \qquad i=1,2\dots n \end{aligned}\right\} \text{--- (2)}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \text{--- (3)}$$

and $x_{ij} \geq 0$ for all $i=1,2\dots m$, $j=1,2\dots n$.

Thus, the transportation problem is a L.P.P of special type, where we are required to find the values of m,n, variable that minimized the objective function Z given by (1), satisfying $(m+n)$ restrictions (2), restrictions (4) and the non negative restriction of variable.

$m \times n$ transportation problem $m+n-1$ equations forms a linearly independent set.